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DEFECT CORRECTIONS VIA NEIGHBOURING PROBLEMS. I. GENERAL THEORY--ETC(U)
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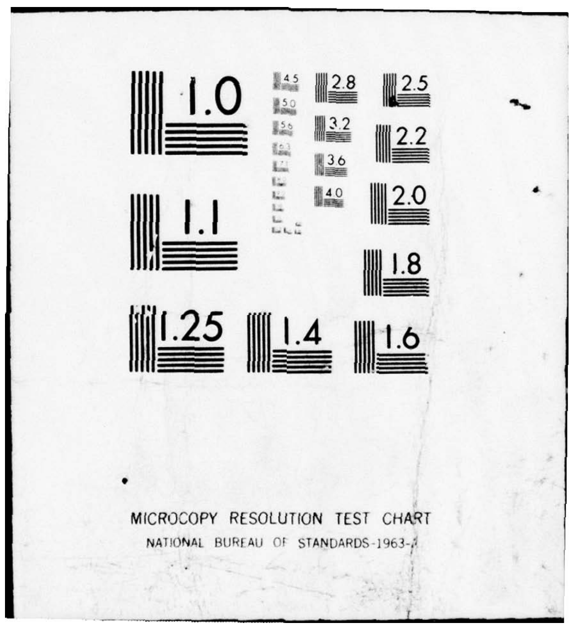
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DEFECT CORRECTIONS VIA NEIGHBOURING PROBLEMS. I. GENERAL THEORY.

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Klaus Böhmer

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ABSTRACT

This document

To solve $Fy = 0$ numerically ~~we use~~ using two different methods, the first of which is sketched already ~~in [3]~~ and the. Second~~ly~~, ~~we~~ introduce ing a neighbouring problem (N.P.) $Fu = d$, $\|d\|$ absolute value of d small, with known solution. ~~We~~ solve the original problem and the N.P. with the "same discretization method". The known error of the N.P. is used as an estimation for the unknown error of the original problem. These procedures are used iteratively and their relations are discussed. In subsequent papers ~~we will apply our~~ the general theory will be applied to some special cases and will discuss relations to collocation methods and to PEREYRA's deferred correction methods [8,9].

AMS (MOS) Subject Classifications: 65B99, 65J05, 65L05, 65L10, 65M99, 65R05

Key Words: Defect correction, Deferred correction, Numerical solution of functional equations (ODES, PDES, IES), Global error estimate

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Defect Corrections via Neighbouring Problems

I. General Theory.

by

Klaus Böhmer

Summary: To solve a functional equation $Fy = 0$ numerically we use two different methods: The first one, scetched already in [3], combines Newton's method with a discretization of the linear problem. Secondly, we introduce a neighbouring problem (N.P.) $Fu = d$, $\|d\|$ small, with known solution. We solve the original problem and the N.P. with the "same discretization method". The known error of the N.P. is used as an estimation for the unknown error of the original problem. Both methods are used iteratively and their relations are discussed. The idea of using N.P. goes back to ZADUNAISKY [15,16,17], again discussed by STETTER [14]. They treated initial value problems of ordinary differential equations where STETTER used our first method, too. The method of N.P. was applied by FRANK, HERTLING, UEBERHUBER [5,6,7] to initial and boundary value problems of ordinary differential equations. In subsequent papers we will apply our general theory to some special cases and will discuss relations to collocation methods and to PEREYRA's deferred corrections [10,11]. We find numerical methods which seem to be as convenient and appropriate than those derived directly (see [5,6,7,14,15,16,17]).

1. Introduction

With Stetters [13] notations, slightly modified, a discretization method $\mathfrak{M}$, applicable to the given problem $\mathfrak{P}$

(1.1) $\quad F(y) = 0, \; F : D \to E^0, \; D \subseteq E, E^0$ Banach spaces,

is an infinite sequence $\{E_h, E_h^0, \Delta_h, \Delta_h^0, \phi_h\}_{h \in \mathbb{H}}$, $\mathbb{H} \subset (0, h_0] \subset \mathbb{R}_+$, $\inf \mathbb{H} = 0$, such that

$$(1.2)\quad \begin{cases} \Delta_h : E \to E_h,\ \Delta_h^{\,o} : E^o \to E_h^{\,o},\ \Delta_h, \Delta_h^o \text{ linear bounded,} \\ \quad E_h, E_h^{\,o} \text{ Banach spaces, } \dim E_h = \dim E_h^{\,o} < \infty \\ E \text{ is continuously imbeded into a Banach space } \hat{E} \text{ and} \\ \lim\limits_{h\to o} \|\Delta_h y\| = \|y\|_{\hat{E}} \text{ for every fixed } y \in E \subseteq \hat{E} \\ \lim\limits_{h\to o} \|\Delta_h z\| = \|z\| \text{ for every fixed } z \in E^o. \\ \phi_h : C_h \to (E_h \to E_h^{\,o}),\ C_h \subseteq (E \to E^o) \text{ and } F \in C_h \text{ for } h \in \mathbb{H}. \end{cases}$$

Here $\|\cdot\|$ means anyone of the norms in E, E^o, E_h, $E_h^{\,o}$.
Applying $\mathfrak{M}$ to problem (1.1) we find the discretization of (1.1) as

$$(1.3)\quad \begin{cases} \Phi_h := \phi_h(F),\ \Phi_h : D_h \to E_h^{\,o},\ D_h \subseteq E_h \text{ for } h \in \mathbb{H}, \\ \Phi_h(\eta_h) = 0. \end{cases}$$

We assume that (1.1) and (1.3) have unique solutions $y \in D$ and $\eta_h \in D_h$ (see [13] pp 12 ff.). Further we use the equivalent notations

$$x_1 = x_2 + O(h^r),\ r \in \mathbb{R}_+ \text{ iff } \|x_1 - x_2\| = O(h^r) \text{ for } h\to 0.$$

If there is a sequence of $\Lambda_h : E \to E^o$, $h \in \mathbb{H}$ such that

$$(1.4)\qquad \Phi_h(\Delta_h u) = \Delta_h^o \{F(u) + \Lambda_h(u)\} \text{ for every } u \in E$$

Λ_h is called a "local error mapping". In many cases the local error mapping Λ_h admits an asymptotic expansion up to the order ν_q, that is

$$(1.5)\quad \begin{cases} \Delta_h^{\,o}\Lambda_h(u) = \Delta_h^{\,o}\{\sum\limits_{\iota=1}^{q} h^{\nu_\iota} f_\iota(u) + O(h^{\nu_{q+1}})\},\ 0<\nu_1<\nu_2<\ldots<\nu_{q+1} \\ \text{for } u \in D_q \subseteq D,\ f_\iota : D_q \to E^o,\ f_\iota \text{ independent of } h. \end{cases}$$

If $y \in D_q$ is the solution of (1.1) and if (1.3) is consistent to (1.1) of order ν_p then the "local discretization error" $\lambda_h := \Delta_h^{\circ}\Lambda_h(y)$ satisfies

$$(1.6) \qquad \lambda_h = \Phi_h(\Delta_h y) = \Delta_h^{\circ}\{\sum_{\iota=p}^{q} h^{\nu_\iota} f_\iota(y) + O(h^{\nu_{q+1}})\} .$$

Very important for numerical applications is the question, whether (1.6) carries over to the "global discretization error"

$$(1.7) \qquad \gamma_h := \eta_h - \Delta_h y,\ \eta_h,\ y \text{ solutions of (1.1), (1.3).}$$

Gragg [8] has studied this question first for initial value problems of ordinary differential equations. Stetter [12,13] generalized Gragg's result to functional equations. In these papers we always have $\nu_\iota = \iota$ or $\nu_\iota = 2\iota$. Several difficult special cases were treated directly (see for example Benson [1]).

2. Asymptotic expansion of the global discretization error

Here we generalize and modify Stetters [12,13] results a little bit to prepare it for our later applications. We require that the ν_ι in (1.5) are such that

$$(2.1) \qquad \begin{cases} \mathbb{N}_* := \{\nu_1,\nu_2,\ldots,\nu_q,\nu_{q+1}\},\ 0<\nu_1<\nu_2<\ldots<\nu_q<\nu_{q+1} \text{ and} \\ \nu_i,\nu_j \in \mathbb{N}_* \text{ implies } \nu_i+\nu_j \in \mathbb{N}_* \text{ or } \nu_i+\nu_j > \nu_{q+1} . \end{cases}$$

For this case we give (see Stetter [13], p. 25).

Definition 2.1: *We call the asymptotic expansion* (1.5) *of the local error mapping* (ν_q,ν_p)*-smooth at* u, *if the derivatives*

$f_\iota^{(\sigma)}(u), \sigma=1,2,\ldots, [\frac{\nu_q-\nu_\iota}{\nu_p}] = \max\{n \in \mathbb{Z} \mid n \leq \frac{\nu_q-\nu_\iota}{\nu_p}\}$ *exist and if*

for $u, e_k \in D_q = D_{\nu_q} \subseteq D$, $\| \sum_{\iota=p}^{q} h^{\nu_\iota} e_\iota \|$ *small enough,* $k=p(1)q$, *the following relation is valid*

$$
(2.2)\quad \begin{cases} \sum_{\iota=1}^{q} h^{\nu_\iota}\{f_\iota(u) + \sum_{\sigma=1}^{[\frac{\nu_q-\nu_\iota}{\nu_p}]} \frac{1}{\sigma!} f_\iota^{(\sigma)}(u)(\sum_{k=p}^{q} h^{\nu_k}e_k)^{\sigma}\} \\ \\ = \Lambda_h(u + \sum_{k=p}^{q} h^{\nu_k}e_k) + O(h^{\nu_{q+1}}) \quad . \end{cases}
$$

In many cases the conditions $u,e_k \in D_q = D_{\nu_q}$ are too stringent for (2.2). So we give

<u>Definition 2.2:</u> *Let* $D_{q,k} \subseteq E_{q,k}$, $k=p(1)q$, *be such that* $D_q \subseteq D_{q,k} \subseteq D$, $D_q \subseteq E_q$, *and that*

$$
(2.3)\quad \begin{cases} \Lambda_h(u + \sum_{k=p}^{q} h^{\nu_k} e_k) \text{ and} \\ \\ f_\iota^{(\sigma)}(u)\ (\sum_{k=p}^{q^*} h^{\nu_k} e_k)^{\sigma}, \quad \iota=1(1)q,\ \sigma=1(1)[\frac{\nu_q-\nu_\iota}{\nu_p}] \end{cases}
$$

with $q^* = q^*(q,p,\iota,\sigma)$ *such that* $\nu_{q^*} = \nu_q-\nu_\iota-(\sigma-1)\nu_p$

are defined for $u \in D_q$, $e_k \in D_{q,k}$. *Further let*

$$
(2.4)\quad \begin{cases} \Lambda_h^*(u + \sum_{k=p}^{q} h^{\nu_k} e_k) := \sum_{\iota=1}^{q} h^{\nu_\iota}\{f_\iota(u) + \\ \\ + \sum_{\sigma=1}^{[\frac{\nu_q-\nu_\iota}{\nu_q}]} \frac{1}{\sigma!} f_\iota^{(\sigma)}(u)\ (\sum_{k=p}^{q^*} h^{\nu_k} e_k)^{\sigma}\} \quad , \end{cases}
$$

satisfy

$$
(2.5)\quad \Lambda_h(u + \sum_{k=p}^{q} h^{\nu_k} e_k) = \Lambda_h^*(u + \sum_{k=p}^{q} h^{\nu_k} e_k) + O(h^{\nu_{q+1}}).
$$

Then the $D_{q,k}$ *are called* <u>*(asymptotically) admissible sets*</u> *(for* $\mathfrak{M}$ *and* $\mathfrak{P}$ *).*

The summation index q^* in (2.4) is chosen such that e_k with $k > q^*$ only contributes to $O(h^{\nu_{q+1}})$. Since further $\nu_p \leqq \nu_{q^*}$ iff $\nu_\iota \leqq \nu_q - \sigma\nu_p$, only those $f_\iota^{(\sigma)}(u)$ contribute to Λ_h^* with $\nu_\iota \leqq \nu_q - \sigma\nu_p$.

Now the following Proposition is simply proved:

<u>Proposition 2.3:</u> *Let the operators given in* (2.3) *be defined and continuous for* $u \in D_q$, $e_k \in D_{q,k}$ *with respect to the norms in* E_q, $E_{q,k}$. *Then* (2.5) *is satisfied.*

Since we are concerned only with asymptotic expansions, $h^{\nu_{q+1}}$-terms are fairly uninteresting. So in addition to ordinary Frechet-derivatives we introduce the following type:

<u>Definition 2.4:</u> Φ_h *is called* r*-times* (ν_q,ν_p)*-differentiable at* η*, if there are bounded* σ*-linear operators* $\Phi_h^{\langle\sigma\rangle}(\eta)$:

$(\mathbb{E}_h)^\sigma \to E_h^0$, *such that for* $\nu_q \geqq r\nu_p$, $e_\iota \in D_{q,\iota}$, $\iota = p(1)q$, *and* $\| \sum_{\iota=p}^{q} h^{\nu_\iota} e_\iota \|$ *small enough*

$$(2.6)\quad \begin{cases} \Phi_h\left(\eta + \Delta_h \sum_{\iota=p}^{q} h^{\nu_\iota} e_\iota\right) - \Phi_h(\eta) \\ \\ = \sum_{\sigma=1}^{r} \frac{1}{\sigma!} \Phi_h^{\langle\sigma\rangle}(\eta)\left(\Delta_h \sum_{\iota=p}^{q} h^{\nu_\iota} e_\iota\right)^\sigma + O(h^{\min\{(r+1)\nu_p,\nu_{q+1}\}}). \end{cases}$$

The operator $\Phi_h^{\langle\sigma\rangle}(\eta)$ *is called the* <u>σ*-th* (ν_q,ν_p)*-derivative of* Φ_h *at* η</u> .

Now the following Proposition is a simple consequence of the last two definitions:

<u>Proposition 2.5:</u> *Let* Λ_h *be* (ν_q,ν_p)*-smooth in* $B_\rho(y):=\{u\in D \mid \|u-y\|<\rho\}$ *and let, with* $\nu_q \geqq r\cdot\nu_p$*, F have r Lipschitz-continuous derivatives. Then* $\phi_h = \Delta_h{}^o\{F+\Lambda_h\}$ *is r-times* (ν_q,ν_p)*-differentiable in* $\Delta_h\, B_\rho(y)$ *and with j such that* $\nu_j<\min\{(r+1)\nu_p,\nu_{q+1}\}-\sigma\nu_p$ *we find*

$$(2.7)\qquad \phi_h{}^{\langle\sigma\rangle}(\Delta_h u) = \Delta_h{}^o\{F^{(\sigma)}(u) + \sum_{\iota=1}^{j} h^{\nu_\iota} f_\iota^{(\sigma)}(u)\},\ \sigma=1(1)r.$$

For $\sigma \leqq r \leqq [\nu_q/\nu_p]$*, with at most one =-sign, we always have* $\min\{(r+1)\nu_p,\nu_{q+1}\}-\sigma\nu_p>\nu_p$*, so* $j \geqq p \geqq 1$ *and the* $\sum$ *in (2.7) is well defined. For* $\sigma = r = [\nu_q/\nu_p]$ *it may be that* $j < 1$*. Then we define* $\sum\limits_{\iota=1}^{j} := 0$ *in (2.7).*

The proof of Proposition 2.5 is obvious with Definitions 2.1 and 2.4 and (2.1).

To give the following Theorem we need some formal notations. Let F be $[\nu_q/\nu_p]$-times differentiable, let y be the solution of (1.1) and let Λ_h be (ν_q,ν_p)-smooth. Then, with $F(y)=0$, $f_\iota(y)=0$, $\iota=1(1)p-1$, y, $e_\iota \in D_q$

$$(2.8)\quad \begin{cases} (F + \Lambda_h)(y + \sum\limits_{\iota=p}^{q} h^{\nu_\iota} e_\iota) + O(h^{\nu_{q+1}}) = \sum\limits_{\iota=p}^{q} h^{\nu_\iota} f_\iota(y) \\ \quad + \sum\limits_{\sigma=1}^{[\nu_q/\nu_p]} \frac{1}{\sigma!}\{F^{(\sigma)}(y) + \sum\limits_{\iota=1}^{\nu_\iota\leqq\nu_q-\sigma\nu_p} h^{\nu_\iota} f_\iota^{(\sigma)}(y)\}(\sum\limits_{k=p}^{q} h^{\nu_k} e_k)^\sigma \end{cases} .$$

Now we introduce operators $g_j(.,\ldots,.)$ by collecting equal powers of h in

$$(2.9)\quad \begin{cases} \sum\limits_{\sigma=2}^{[\nu_q/\nu_p]} \frac{1}{\sigma!}\{F^{(\sigma)}(y) + \sum\limits_{\iota=1}^{\nu_\iota\leqq\nu_q-\sigma\nu_p} h^{\nu_\iota} f_\iota^{(\sigma)}(y)\}(\sum\limits_{k=p}^{q} h^{\nu_k} e_k)^\sigma \\ = \sum\limits_{j=p_1}^{q} h^{\nu_j} g_j(e_p,\ldots,e_{\ell(j)}),\ g_j = 0,\ j=p(1)p_1-1, \\ \text{with } \nu_{p_1} = 2\nu_p,\ \nu_{\ell(j)} = \nu_j - \nu_p, \text{ therefore } \ell(j) < j; \end{cases}$$

in the first line of (2.9) e_k gives contributions to powers $\geq \nu_p + \nu_k$. So the biggest k appearing in $g_j(e_p,\ldots,e_k)$ is $k=\ell(j)$ with $\nu_j = \nu_p + \nu_{\ell(j)}$. For the special case $\nu_\iota=\iota$, $\iota=1(1)q$, we have $\ell(j) = j - p$. With the g_j, defined in (2.9), we finally have

$$
(2.10)\quad \begin{cases} (F + \Lambda_h)(y + \sum_{\iota=p}^{q} h^{\nu_\iota} e_\iota) + O(h^{\nu_{q+1}}) = \sum_{j=p}^{q} h^{\nu_j}\{F'(y)e_j + f_j(y) \\[2ex] \quad + \sum_{\substack{\iota\geq 1,k\geq p \\ \nu_\iota+\nu_k=\nu_j}} f_\iota'(y)e_k + g_j(e_p,\ldots,e_{\ell(j)})\},\ \nu_j = \nu_p + \nu_{\ell(j)}. \end{cases}
$$

Since we want to have

$$\eta = \Delta_h(y + \sum_{\iota=p}^{q} h^{\nu_\iota} e_\iota) + O(h^{\nu_{q+1}})$$

and since

$$\Phi_h(\Delta_h u) = \Delta_h^{O}\{F(u) + \Lambda_h(u)\}$$

the proof of the following Theorem mainly consists in equating to zero all the coefficients of h^{ν_ι}, $\iota=p(1)q$, and can be done exactly in the same way like in Stetter [13]. (Due to $\mathbb{N}$: $\nu_p([\nu_q/\nu_p]+1)\geq\nu_{q+1}$!)

<u>Theorem 2.6:</u> *Let the original problem* $\mathcal{P}$ *with the exact solution* y *and the discretization method* $\mathcal{M}$ *, applicable to* $\mathcal{P}$ *, satisfy*

(2.11)
- (a) $\mathcal{M}$ *is stable for* $\mathcal{P}$; (see [13]);
- (b) $\mathcal{M}$ *is consistent with* $\mathcal{P}$ *of order* ν_p;
- (c) *an asymptotic expansion* (1.5) *of the local error mapping* Λ_h *exists and* $y \in D_q$;
- (d) Λ_h *is* (ν_q,ν_p)*-smooth at* y;
- (e) F *has* $[\nu_q/\nu_p]$ *Lipschitz-continuous bounded derivatives in* $B_\rho(y)$;
- (f) $(F'(y))^{-1}$ *exists and is bounded.*

Define the e_j, $j = p(1)q$ *by* (see (2.9), (2.10))

$$(2.12)\quad F'(y)e_j = -\{f_j(y) + \sum_{\substack{\iota\geq 1, k\geq p\\ \nu_\iota+\nu_k=\nu_j}} f'_\iota(y)\, e_k + g_j(e_p,\ldots,e_{\ell(j)})\},$$

and let $e_j \in D_{q,j}$.

Then the global discretization error $\gamma_h = \eta_h - \Delta_h y$ *admits a unique asymptotic expansion*

$$(2.13)\qquad \eta_h - \Delta_h y = \Delta_h \sum_{\iota=p}^{q} h^{\nu_\iota} e_\iota + O(h^{\nu_{q+1}}) .$$

For the rest of the paper we postulate (2.13).

For our later applications we need the structure and the arguments of the g_j in (2.9) a little bit more completely:

Proposition 2.7: *The* g_j *in* (2.9) *are operators acting multilinearly on the* e_k, $k=p(1)\ell(j)$, $\nu_j = \nu_p + \nu_{\ell(j)}$, *and linearly on the* $F^{(\sigma)}(y)$, $\sigma=2(1)[\nu_j/\nu_p]$, *and the* $f_\iota^{(\sigma)}(y)$, $\sigma=2(1)[\frac{\nu_j-\nu_\iota}{\nu_p}]$. $f_\iota^{(\cdot)}(y)$ *contributes to* g_j *iff* $2 \leq [\frac{\nu_j-\nu_\iota}{\nu_p}]$ *or for* $\iota=2(1)\iota(j)$ *with* $\nu_{\iota(j)} \leq \nu_j-2\nu_p < \nu_{\iota(j)+1}$. *So*

$$(2.14)\quad \left\{\begin{array}{l} g_j=g_j(e_p,\ldots,e_{\ell(j)};F''(y),\ldots,F^{([\nu_j/\nu_p])}(y);f''_1(y),\\[2ex] \ldots,f_1^{([\frac{\nu_j-\nu_1}{\nu_p}])}(y),f''_\iota(y),\ldots,f_\iota^{([\frac{\nu_j-\nu_\iota}{\nu_p}])}(y),\ldots,f''_{\iota(j)}(y)). \end{array}\right.$$

Here $F^{(\sigma)}(y)$ *resp.* $f_\iota^{(\sigma)}(y)$ *has to be applied to those* e_k *with*
(2.15) $\nu_k \leq \nu_j-(\sigma-1)\nu_p$ *resp.* $\nu_k \leq \nu_j - \nu_\iota - (\sigma-1)\nu_p$.

Proof: We only give the arguments for the $f_\iota^{(\sigma)}$. By (2.9) we have contributions to $h^{\nu_j} g_j$ by $f_\iota^{(\sigma)}(y)(\sum_{k=p}^{q} h^{\nu_k} e_k)^\sigma$ if $\nu_\iota + \sigma\nu_p \leqq \nu_j$, that is for $\sigma = 2(1)[\frac{\nu_j - \nu_\iota}{\nu_p}]$. From $2 \leqq [\frac{\nu_j - \nu_\iota}{\nu_p}]$ we get $\iota(j)$. □

3. Defect corrections via Newton's method

The results given here are slight modifications and extensions of those given in [3]. We use an operator (see for example [9])

$$(3.1) \qquad T_\delta : \begin{cases} E_h \to E \\ \\ \eta_h \mapsto y_h := T_\delta \eta_h \end{cases}, \quad \delta \in \mathbb{R}_o := \{t \in \mathbb{R} \mid t \geqq 0\}$$

which essentially reproduces the asymptotic expansion of η_h in (2.13):

$$(3.2) \qquad \begin{cases} \text{If } \eta_h \text{ satisfies (2.13) then} \\ T_\delta \eta_h = y_h = y + \sum_{\iota=p}^{\overline{q}} h^{\nu_\iota} e_\iota(y) + O(h^{\nu_{\overline{q}}^{*}}), \\ \text{where } \nu_{\overline{q}} \leqq \nu_q - \delta < \nu_{\overline{q}}^{*} \leqq \nu_{\overline{q}+1} . \end{cases}$$

Usually T_δ are interpolation operators and often $\|\cdot\|_E$ includes derivatives, so one has $\overline{q} \leqq q$. We will have to compute the defect $F(y_h) = F(T_\delta \eta_h)$ and we need an asymptotic expansion

$$(3.3) \qquad F(y_h) = \sum_{\iota=p}^{\overline{q}} h^{\nu_\iota} g_\iota(y) + O(h^{\nu_{\overline{q}}^{*}})$$

with the $\overline{q}$ from (3.2) (see (3.7)). Sometimes, for example in spline interpolation, $y_h = T_\delta \eta_h$ is not smooth enough to admit the full expansions in (3.2) resp. (3.3). In those cases we have to cut down the asymptotic expansion (2.13) to

$$\eta_h - \Delta_h y = \Delta_h \sum_{\iota=p}^{q_o} h^{\nu_\iota} e_\iota + O(h^{\nu_{q_o+1}}) \text{ with } q_o < q$$

such that this expansion admits (3.2), (3.3).

As a consequence of the modified Newton method we find [2]

Theorem 3.1: *Let* $y_o \in D$ *be an approximation for the solution* y *of* (1.1) *such that* F *is at least* $K := [\nu_q/\nu_p]$*-times differentiable with uniformly Lipschitz-continuous derivatives in* $B_\rho(y_o)$ *and let* $F^*(y_o) \in \mathcal{L}(E,E^o)$ *be invertible with*

$$\|(F^*(y_o))^{-1}\| \leq \mu, \qquad \mu \cdot \sup_{z \in B_\rho(y_o)} \|F'(z) - F^*(y_o)\| \leq q < 1$$

$$\|F^*(y_o)^{-1} F(y_o)\| \leq \rho \cdot (1-q) .$$

Then the modified Newton method

$$\text{(3.4)} \quad \left\{ \begin{array}{ll} F^*(y_o)(y_\ell - y_{\ell-1}) = -F(y_{\ell-1}) , & \ell = 1,2,3,\ldots , \\ \textit{with } \nu_{p_o} := \nu_p,\ \nu_{p_{\ell-1}} + \nu_{p_o} = \nu_{p_\ell} , & \ell = 1,2,3 \end{array} \right.$$

defines a sequence y_ℓ *converging to* y, $\lim_{\ell\to\infty} \|y_\ell - y\|_E = 0$. *If further* $F'(y)^{-1}$ *exists and is bounded and*

$$\text{(3.5)} \quad \left\{ \begin{array}{l} y_o = y + \sum_{\iota=p_o}^{q} h^{\nu_\iota} e_\iota(y) + O(h^{\nu_{q+1}}),\ e_{\iota,o}(y) := e_\iota(y), \\ F^*(y_o) = F'(y) + \sum_{\iota=p}^{\hat{q}} h^{\nu_\iota} C_\iota(y) + O(h^{\nu_{\hat{q}+1}}),\ \nu_{\hat{q}} + \nu_p = \nu_q \\ C_\iota(y) \in \mathcal{L}(E,E^o),\ C_\iota(y) \textit{ independent of } h, \end{array} \right.$$

then y_ℓ *admits an asymptotic expansion of the form*

$$
(3.6)\qquad \begin{cases} y_\ell = y + \sum_{\iota=p_\ell}^{q} h^{\nu_\iota} e_{\iota,\ell}(y) + O(h^{\nu_{q+1}}) \ \textit{for}\ p_\ell \leqq q \\[2ex] \textit{resp.}\ y_\ell = y + O(h^{\nu_{q+1}}) \ \textit{for}\ p_\ell > q\,. \end{cases}
$$

If $F^*(y_o) = F'(y_o)$, *the second part of* (3.5) *follows from the Taylor-formula.*

If y_o *and* $F^*(y_o)$ *satisfy* (3.5) *then we have*

$$
F^*(y_o) = F'(y_o) + \sum_{\iota=p}^{\hat q} h^{\nu_\iota} \tilde{C}_\iota(y) + O(h^{\nu_{\hat q+1}}),
$$

$$
\tilde{C}_\iota(y) \in \mathcal{L}(E,E^o).
$$

<u>Proof:</u> The convergence of the modified Newton method under the restrictions on y_o and $F^*(y_o)$ follows by usual conclusions (see [2]). (3.6) is proved by induction. Let $y_{\ell-1}$ be of the form (3.6) with p_ℓ and $e_{\iota,\ell}$ replaced by $p_{\ell-1}$ and $e_{\iota,\ell-1}$. Then we have with Taylor's formula and because of the Lipschitz-continuous $F^{(\kappa)}$ - we suppress the argument y in the asymptotic expressions -

$$
F(y_{\ell-1}) = F(y + \sum_{\iota=p_{\ell-1}}^{q} h^{\nu_\iota} e_{\iota,\ell-1} + O(h^{\nu_{q+1}}))
$$

$$
= F(y) + (F^*(y_o) + F'(y) - F^*(y_o)) \sum_{\iota=p_{\ell-1}}^{q} h^{\nu_\iota} e_{\iota,\ell-1} + O(h^{\nu_{q+1}})
$$

$$
+ \sum_{j=2}^{\kappa} \frac{F^{(j)}(y)}{j!} \Big(\sum_{\iota=p_{\ell-1}}^{q} h^{\nu_\iota} e_{\iota,\ell-1} \Big)^j + O(h^{(\kappa+1)\nu_{p_{\ell-1}}}).
$$

Since $(\kappa+1)\nu_{p_{\ell-1}} \geqq (\kappa+1)\nu_p = \kappa\,\nu_p + \nu_p \geqq \nu_{q+1}$ and with $F(y) = 0$ and (3.5) we find

$$
(3.7)\qquad F(y_{\ell-1}) = F^*(y_o) \sum_{\iota=p_{\ell-1}}^{p_\ell-1} h^{\nu_\iota} e_{\iota,\ell-1} + \sum_{\iota=p_\ell}^{q} h^{\nu_\iota} \tilde{e}_{\iota,\ell} + O(h^{\nu_{q+1}})
$$

with $e_{\iota,\ell-1}$ and $\tilde{e}_{\iota,\ell}$ independent of h. Now (3.4), (3.5) and (3.7) imply

$$y_\ell = y + \sum_{\iota=p_\ell}^{q} h^{\nu_\iota} (e_{\iota,\ell-1} - F^*(y_o)^{-1} \tilde{e}_{\iota,\ell}) + O(h^{\nu_{q+1}}) .$$

Again (3.5) implies

$$F^*(y_o)^{-1} = F'(y)^{-1} (id_{E^o} + \sum_{\iota=p}^{\hat{q}} h^{\nu_\iota} C_\iota^* + O(h^{\nu_{\hat{q}+1}}))$$

with C_ι^* independent of h, and a combination of the two last results proves (3.6). The last statement again follows with the Taylor formula. □

Since, again, (3.4) is usually not solvable exactly, we have to use a discretization method. The crucial question is, if this discretization method reproduces the asymptotic expansion of the y_ℓ. To guarantee that we need

Definition 3.2: *In* (1.1) *let* $Fu=F_1u+d$ *resp.* $Fu=F_1u+F_2u$ *with* $F_1, F_2 \in \mathscr{L}(E,E^o)$ *and* d *independent of* u. *Let* $\mathfrak{M}$, *applied to* $\mathscr{P}$, *give*

$$(3.8) \begin{cases} \Phi_h(F_1u+d)=\Phi_h(F_1)\Delta_h u+\Delta_h^o d \text{ with } \Phi_h(F_1),\Phi_h(F_2) \in \mathscr{L}(E_h,E_h^o) \text{ resp.} \\ \Phi_h(F_1u+F_2u)=\Phi_h(F_1)\Delta_h u+\Phi_h(F_2)\Delta_h u \ . \end{cases}$$

Then we call $\mathfrak{M}$ *a* *linear discretization method* *for* $\mathscr{P}$ *and* $\Phi_h(F_1)\Delta_h u+\Delta_h^o d =0$ *resp.* $\Phi_h(F_1)\Delta_h u+\Phi_h(F_2)\Delta_h u=0$ *linear discretizations* *of* $F_1u+d=0$ *resp.* $F_1u+F_2u=0$.

Let

$$(3.9)\quad \begin{cases} y_{\ell-1} := T_\delta \eta_{h,\ell-1}\ , \quad \ell=1,2,\dots \text{ and} \\ \\ \Phi_h^*(\Delta_h y_o)(\eta_{h,\ell} - \eta_{h,\ell-1}) = -\Delta_h^o F(y_{\ell-1}) \end{cases}$$

be a linear discretization of (3.4). Sometimes we use $F^*(y_o)=F'(y_o)$, then often Φ_h^* is the Frechet derivative of Φ_h from (1.3), but that is not true in every case. We want to prove a result for the $\eta_{h,\ell}$ in (3.9) similar to that for the y_ℓ in (3.6). Since we have to use in (3.9) the $T_\delta\ \eta_{h,\ell-1}$ not only the lower index p_ℓ, but also the upper index q has to be changed for every step corresponding to T_δ, and we define (see (3.2) and (3.4))

$$(3.10)\qquad \nu_{q_\ell} \leqq \nu_{q_{\ell-1}} - \delta < \nu_{q_\ell}^* \leqq \nu_{q_\ell+1}\ .$$

Now the following Theorem is valid

Theorem 3.3: *In addition to the conditions in Theorem 3.1 let* $\Phi_h^*(\Delta_h y_o)\Delta_h u = \Delta_h^o d$ *be a stable linear discretization for* $F^*(y_o)u=d$. *Further let in* (3.9) *the* $\eta_{h,\ell-1}$ *and therefore the corresponding defect* $F(y_{\ell-1})$

$$(3.11)\quad \begin{cases} \eta_{h,\ell-1} = \Delta_h\{y + \sum\limits_{\iota=p_{\ell-1}}^{q_{\ell-1}} h^{\nu_\iota} e_{\iota,\ell-1}(y)\} + O(h^{\nu^*_{q_{\ell-1}}})\ , \quad \ell=1,2,\dots \\ \\ F(y_{\ell-1}) = F^*(y_o)\{\sum\limits_{\iota=p_{\ell-1}}^{p_\ell-1} h^{\nu_\iota} e_{\iota,\ell-1}(y)\} + \sum\limits_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota}\tilde{e}_{\iota,\ell}(y) + O(h^{\nu^*_{q_\ell}}) \end{cases}$$

be such that (with $e_{\iota,\ell-1}, \tilde{e}_{\iota,\ell}, k_{\iota,\ell-1,i}, e^+_{\iota,\ell}, g_{\iota,\ell,i}$ *independent of* h*)*

$$(3.12)\quad \begin{cases} \Phi_h^*(\Delta_h y_o)\varepsilon_{\iota,\ell-1} = \Delta_h^o F^*(y_o)e_{\iota,\ell-1} \textit{ implies} \\ \varepsilon_{\iota,\ell-1} = \Delta_h\{e_{\iota,\ell-1} + \sum_{i=p_o}^{s_{\iota,\ell-1}} h^{\nu_i} k_{\iota,\ell-1,i}\} + O(h^{\nu^*_{s_{\iota,\ell-1}}}) \\ \textit{with } \nu_\iota + \nu_{s_{\iota,\ell-1}} = \nu_{q_{\ell-1}},\ \nu_\iota + \nu_{s_{\iota,\ell-1}+1} \geqq \nu^*_{q_{\ell-1}} \textit{ and} \\ \Phi_h^*(\Delta_h y_o)\tilde{\varepsilon}_\iota = \Delta_h^o \tilde{e}_{\iota,\ell} \textit{ implies } \tilde{\varepsilon}_\iota = \Delta_h\{e^+_{\iota,\ell} + \sum_{\iota=p_o}^{s_{\iota,\ell-1}} h^{\nu_\iota} g_{\iota,\ell,i}\} + \\ \qquad + O(h^{\nu^*_{s_{\iota,\ell-1}}}) . \end{cases}$$

Then, for h small enough, $\eta_{h,\ell}$ *from* (3.9) *satisfies*

$$(3.13)\qquad \eta_{h,\ell} = \Delta_h\{y + \sum_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota} e_{\iota,\ell}(y)\} + O(h^{\nu^*_{q_\ell}}) \textit{ for } p_\ell \leqq q_\ell$$

resp.

$$(3.14)\qquad \eta_{h,\ell} = \Delta_h y + O(h^{\nu^*_{q_\ell}}) \textit{ for } p_\ell > q_\ell .$$

Proof: In a totally analogous way to (3.7) we find from (3.2), (3.10), (3.11) and

$$y_{\ell-1} = T_\delta \eta_{h,\ell-1} = y + \sum_{\iota=p_{\ell-1}}^{q_\ell} h^{\nu_\iota} e_{\iota,\ell-1}(y) + O(h^{\nu^*_{q_\ell}})$$

that

$$(3.15)\qquad F(y_{\ell-1}) = F^*(y_o)\{\sum_{\iota=p_{\ell-1}}^{p_\ell-1} h^{\nu_\iota} e_{\iota,\ell-1}(y)\} + \sum_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota} \tilde{e}_{\iota,\ell}(y) + O(h^{\nu^*_{q_\ell}}).$$

With the stability, (3.9) and (3.12) we conclude

$$\eta_{h,\ell} - \eta_{h,\ell-1} = -\Delta_h \{\sum_{\iota=p_{\ell-1}}^{p_\ell-1} h^{\nu_\iota} e_{\iota,\ell-1}(y) + \sum_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota} \hat{e}_{\iota,\ell}(y)\} + O(h^{\nu^*_{q_\ell}})$$

or finally with (3.11)

$$\eta_{h,\ell} = \Delta_h\{y + \sum_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota} e_{\iota,\ell}(y)\} + O(h^{\nu^*_{q_\ell}}) \text{ for } p_\ell \leqq q_\ell$$

resp. (3.14) if $p_\ell > q_\ell$. □

To find (3.13) or (3.14) one has to know $F(y_{\ell-1})$ exactly. In many applications that is not the case. This difficulty is overcome in the following

<u>Theorem 3.4:</u> *Define the* $\eta^*_{h,\ell}$ *by* $\eta^*_{h,o} = \eta_{h,o}$ *and*

$$(3.16) \qquad \Phi_h^*(\Delta_h y_o)(\eta^*_{h,\ell} - \eta^*_{h,\ell-1}) = -\psi(\eta^*_{h,\ell-1}), \quad \ell=1,2,\ldots$$

where

$$(3.17) \qquad \psi(\eta^*_{h,\ell-1}) = \Delta_h^o\{F(T_\delta\eta^*_{h,\ell-1}) + \sum_{\iota=p_\ell}^{q_{\ell-1}} h^{\nu_\iota} f_{\iota,\ell-1}(y)\} + O(h^{\nu^*_{q_{\ell-1}}}).$$

Further let the conditions of Theorem 3.3 be satisfied with $\eta_{h,\ell-1}, e_{\iota,\ell-1}$ *a.s.o. in* (3.11), (3.12) *replaced by* $\eta^*_{h,\ell-1}, e^*_{\iota,\ell-1}$ *a.s.o. Then* (3.13) *resp* (3.14) *are valid for* $\eta^*_{h,\ell}$ *instead of* $\eta_{h,\ell}$.

<u>Proof:</u> The proposition follows by comparing (3.15) and (3.17). □

The question, how exactly $\eta_{h,o} = \eta_h$ in $\Phi_h(\eta_h) = 0$ should be known, may be treated exactly in the same way as it is done in [10] by introducing the concept of approximate solutions.

4. Defect corrections via neighbouring problems

If the discretization (1.3) of the original problem (1.1) is good and if h is small enough then $y_o := T_\delta\eta_{h,o}$ (see (3.1), (3.2), (3.9)) will be a good approximation for the solution y of (1.1) and the

defect $d_o := F(y_o)$ will be small. We assume that the neighbouring problems (N.P.)

$$(4.1) \qquad F(u) = d \;, \; \|d\| < c_o \in \mathbb{R}_+ \quad \text{N.P. for (1.1)}$$

and, by discretization of (4.1),

$$(4.2) \qquad \Phi_h(\xi_h) = \Delta_h^o d, \; \|d\| < c_o \in \mathbb{R}_+ \; \text{N.P. for (1.3)}$$

are uniquely solvable. So y_o is the unique solution of

$$(4.3) \qquad F(u) = F(y_o), \text{ if } \|F(y_o)\| < c_o \; .$$

Though we know the exact solution y_o of (4.3) we use the corresponding discretization to compute $\xi_{h,o}$ from

$$(4.4) \qquad \Phi_h(\xi_{h,o}) = \Delta_h^o F(y_o) \; .$$

If $\|F(y_o)\|$ is small enough the known error

$$(4.5) \qquad \xi_{h,o} - \Delta_h y_o = \xi_{h,o} - \eta_{h,o} + O(h^{\nu_{q_o}^*}) \; ,$$

see (3.2), (3.9), should be a good estimation for the unknown error $\eta_{h,o} - \Delta_h y$ of the original problem (1.1) and of its discretization (1.3) and

$$(4.6) \qquad \eta_{h,1} := \eta_{h,o} - (\xi_{h,o} - \Delta_h y_o) = \eta_{h,o} - (\xi_{h,o} - \eta_{h,o}) + O(h^{\nu_{q_o}^*})$$

should be a better approximation than $\eta_{h,o}$ and the defect $F(y_1)$ with $y_1 := T_\delta \eta_{h,1}$, should be smaller than $F(y_o)$. ($\eta_{h,1}$ in (4.6) is usually different from $\eta_{h,1}$ in (3.9).) So

$$F(u) = F(y_1) \quad , \quad \Phi_h(\xi_{h,1}) = \Delta_h^{o} F(y_1)$$

are closer to (1.1), (1.3) than (4.3), (4.4) and the known error

$$\xi_{h,1} - \Delta_h y_1 = \xi_{h,1} - \eta_{h,1} + O(h^{\nu_{q_1}^*})$$

should be a better estimation for the error $\eta_{h,o} - \Delta_h y$ than (4.5) and, corresponding to (4.6), we define

$$\eta_{h,2} := \eta_{h,o} - (\xi_{h,1} - \Delta_h y_1) = \eta_{h,o} - (\xi_{h,1} - \eta_{h,1}) + O(h^{\nu_{q_1}^*})$$

This process may be used to generate the following iteration method

$$(4.7) \qquad \begin{cases} \Phi_h(\eta_{h,o}) = o \\ \left.\begin{array}{l} y_{\ell-1} := T_\delta\, \eta_{h,\ell-1} \\ \Phi_h(\xi_{h,\ell-1}) = \Delta_h^{o}\, F(y_{\ell-1}) \\ \eta_{h,\ell} := \eta_{h,o} - (\xi_{h,\ell-1} - \eta_{h,\ell-1}) \end{array}\right\} \quad \ell = 1,2,3,\ldots\ . \end{cases}$$

Since we hope that the $y_{\ell-1}$ will approach the exact solution y, the defect $F(y_{\ell-1})$ will grow smaller and smaller, so $\Phi_h(\xi_{h,\ell-1}) = \Delta_h F(y_{\ell-1})$ will differ less and less from $\Phi_h(\eta_{h,o}) = o$. Therefore a very good

way to compute $\xi_{h,\ell-1}$ from (4.7) is to use a modified Newton method with starting value $\zeta_o = \eta_{h,o}$

$$(4.8)\quad \begin{cases} \Phi_h'(\eta_{h,o})(\zeta_i - \zeta_{i-1}) = -\{\Phi_h(\zeta_{i-1}) - \Delta_h^o F(y_{\ell-1})\}, \\ i=1,2,\dots;\ \zeta_o = \eta_{h,o},\ \Phi_h(\zeta_o) = \Phi(\eta_{h,o}) = o, \\ \text{and hopefully } \lim_{i\to\infty} \zeta_i = \xi_{h,\ell-1}. \end{cases}$$

As a consequence of (2.6) and (3.3) we will have $\zeta_i - \zeta_{i-1} = \sum_{\ell=p_i}^{q_i} h^{\nu_\ell} g_\ell(y) + O(h^{\nu_{q_i^*}})$ with appropriate $g_\ell, p_i, q_i \leqq q$. Similarly as in 3., we may use in (4.8) instead of $\Phi_h'(\eta_{h,o})$ any $\Phi_h^*(\eta_{h,o})$ with

$$(4.9)\quad \begin{cases} \Phi_h^*(\eta_{h,o}) = \Phi_h'(\eta_{h,o}) + \sum_{\iota=p}^{\hat{q}} h^{\nu_\iota} \Phi_\iota(\eta_{h,o}) + O(h^{\nu_{\hat{q}+1}}), \\ \Phi_\iota \in \mathcal{L}(E_h, E_h^o), \quad \iota=p(1)q,\ \nu_{\hat{q}} + \nu_p = \nu_q. \end{cases}$$

If we do so and compute only the first step in (4.8) we find the following simplified version of (4.7)

$$(4.10)\quad \begin{cases} \Phi_h(\eta_{h,o}) = 0 \\ y_{\ell-1} := T_\delta\, \eta_{h,\ell-1} \qquad\qquad \ell=1,2,\dots \\ \Phi_h^*(\eta_{h,o})(\xi_{h,\ell-1} - \eta_{h,o}) = \Delta_h^o F(y_{\ell-1}) \\ \eta_{h,\ell} := \eta_{h,o} - (\xi_{h,\ell-1} - \eta_{h,\ell-1}) = \eta_{h,\ell-1} - (\xi_{h,\ell-1} - \eta_{h,o}) . \end{cases}$$

To be able to prove the following Theorems we need

Definition 4.1: *Let* $\mathfrak{M}$ *be* (ν_q,ν_p)*-smooth for* $\mathcal{P}$ *. Further let, for* $u \in B_\rho(y)$,

(4.11) $$\Phi_h'(\Delta_h u)\tau = \Delta_h^{o} d$$

be such that the discretization of

(4.12) $$F'(u)v = d$$

by $\mathfrak{M}$

(4.13) $$\phi_h(F'(u))\tau = \Delta_h^{o} d$$

satisfies

(4.14) $$\phi_h(F'(u)) = \Phi_h'(\Delta_h u) + \sum_{\iota=p}^{q} h^{\nu_\iota}\,\phi_\iota(\Delta_h u) + O(h^{\nu_{q+1}}) ,$$

with $\phi_\iota \in \mathcal{L}(E_h,E_h^{o})$ *independent of* h *resp.*

(4.15) $$\phi_h(F'(u)) = \Phi_h'(\Delta_h u) + O(h^{\nu_{q+1}}) .$$

Then $\mathfrak{M}$ *is called a* <u>*differentiable*</u> *resp. a* <u>*strongly differentiable discretization method*</u> *for* $\mathcal{P}$.

If, in addition to (4.9), we have

(4.16) $$F^{*}(u) = F'(u) + \sum_{\iota=p}^{\hat q} h^{\nu_\iota}\,F_\iota(u) + O(h^{\nu_{\hat q+1}}),\ F_\iota \in \mathcal{L}(E,E^{o})$$

then we have an analogous formula to (4.14) with $\phi_h(F'(u))$ and $\Phi_h'(\Delta_h u)$ replaced by $\phi_h(F^{*}(u))$ and $\Phi_h^{*}(\Delta_h u)$. For most of our later applications we have the situation given in

Definition 4.2: *Let* $\Phi_h^{*}(\Delta_h u)$ *in* (4.9) *and* $F^{*}(u)$ *in* (4.16) *be such that, for* d *and* u *smooth enough, the solutions* τ *and* v *of*

(4.17) $$\Phi_h^{*}(\Delta_h u)\tau = \Delta_h^{o} d \text{ and } F^{*}(u)v = d$$

satisfy

$$(4.18) \qquad \tau = \Delta_h\{v + \sum_{\iota=p}^{q} h^{\nu_\iota} f_\iota(v) + O(h^{\nu_{q+1}})\} .$$

Then $(\Phi_h^{*}(\Delta_h u), F^{*}(u))$ *is called* <u>$\mathfrak{M}$*-admissible*</u> *(* $\mathfrak{M}$ *a* (ν_q,ν_p)*-smooth discretization method).*

It is possible, to give sufficient conditions, depending on d,u and the Φ_ι in (4.9) and the F_ι in (4.16), for the $\mathfrak{M}$ -admissibility of $(\Phi_h^{*}(u), F^{*}(u))$. But since these conditions are very complicated and do not save too much work in the special cases to be treated later on, we do not formulate the corresponding theorem. In our later applications we will often use

$$\Phi_h^{*}(\Delta_h u) = \Phi_h'(\Delta_h u) \text{ or } \Phi_h^{*}(\Delta_h u) = \Phi_h^{\langle 1\rangle}(\Delta_h u).$$

If $\mathfrak{M}$ is strongly differentiable for $\mathfrak{P}$, then $(\Phi_h'(\Delta_h u), F'(u))$ is $\mathfrak{M}$ -admissible.

<u>Theorem 4.3:</u> *In addition to the conditions of* Theorem 2.4 *and* 3.1 *let* $\mathfrak{M}$ *be a linear discretization for* $\mathfrak{P}$, $(\Phi_h^{*}(\eta_{h,o}), F^{*}(y_o))$ *be* $\mathfrak{M}$*-admissible and* $\Theta = \{E_h, E_h^{o}, \Phi_h^{*}(\eta_{h,o})\}_{h\in\mathbb{H}}$ *be stable (see* [13] *or* [10]*), that is for this case*

$$\|\gamma\| \leqq c\|\Phi_h^{*}(\eta_{h,o})\gamma\| \textit{ for all } \gamma \in E_h \textit{ and a fixed } c \in \mathbb{R}_+ .$$

Further let (4.2) *be the* $\mathfrak{M}$*-discretization of* (4.1) *and let the* N.P.s (4.1), (1.1) *resp.*(4.2), (1.3) *be uniquely solvable for* $|d| < c_o \in \mathbb{R}_+$. *Finally let the implication* (3.11), (3.12) *be satisfied with* $\Phi_h^{*}(\Delta_h y_o)$ *replaced by* $\Phi_h^{*}(\eta_{h,o})$.

Then, for h *small enough, the* $\eta_{h,\ell}$ *from* (4.10) *satisfy*

$$(4.19) \qquad \eta_{h,\ell} = \Delta_h\{y + \sum_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota} e_{\iota,\ell}(y) + O(h^{\nu^{*}_{q_\ell}}) \textit{ for } p_\ell \leqq q_\ell$$

resp.

$$(4.20)\qquad \eta_{h,\ell} = \Delta_h y + O(h^{\nu^*_{q_\ell}}) \ \textit{for}\ p_\ell > q_\ell \ .$$

<u>Proof:</u> For h small enough, $T_\delta \eta_{h,o}$, and by induction, $T_\delta \eta_{h,\ell-1} \in B_\rho(y)$. Analogously to (3.7) and (3.15) we have

$$(4.21)\qquad \begin{cases} F(y_{\ell-1}) = h^{\nu_{p_{\ell-1}}} F^*(y_o)\{\sum\limits_{\iota=p_{\ell-1}}^{p_\ell-1} h^{\nu_\iota-\nu_{p_{\ell-1}}} e_{\iota,\ell-1}(y)\} \\ \\ \quad + h^{\nu_{p_{\ell-1}}} \sum\limits_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota-\nu_{p_{\ell-1}}} \tilde{e}_{\iota,\ell}(y) + O(h^{\nu^*_{q_\ell}}) \ . \end{cases}$$

Since $F^*(y_o)^{-1}$ exists, the problem

$$F^*(y_o)v = F(y_{\ell-1})$$

is uniquely solvable and we find, by (4.21) and arguments similar to 3.,

$$(4.22)\qquad \begin{cases} v = h^{\nu_{p_{\ell-1}}}\{\sum\limits_{\iota=p_{\ell-1}}^{p_\ell-1} h^{\nu_\iota-\nu_{p_{\ell-1}}} e_{\iota,\ell-1}(y) + \sum\limits_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota-\nu_{p_{\ell-1}}} \hat{e}_{\iota,e}(y)\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + O(h^{\nu^*_{q_\ell}}) \\ \\ \quad = y_{\ell-1} - y + \sum\limits_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota} e^+_{\iota,\ell}(y) + O(h^{\nu^*_{q_\ell}}) \ . \end{cases}$$

Now $(\Phi_h{}^*(\eta_{h,o}), F^*(y_o))$ is $\mathfrak{M}$-admissible, $\mathfrak{M}$ is linear and $\theta = \{E_h, E_h{}^o, \Phi_h{}^*(\eta_{h,o})\}$ stable, so we find with $\tau := \xi_{h,\ell-1} - \eta_{h,o}$ from (4.10), (4.17), (4.18), (4.21), (4.22) that

$$(4.23)\qquad \xi_{h,\ell-1} - \eta_{h,o} = \tau = h^{\nu_{p_{\ell-1}}} \Delta_h\{\sum_{\iota=p_{\ell-1}}^{p_\ell-1} h^{\nu_\iota-\nu_{p_{\ell-1}}} e_{\iota,\ell-1}(y) +$$

$$+ \sum_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota-\nu_{p_{\ell-1}}} e^+_{\iota,\ell}\} + O(h^{\nu^*_{q_\ell}})$$

and so finally

$$(4.24)\quad \begin{cases} \eta_{h,\ell} = \eta_{h,\ell-1} + (\xi_{h,\ell-1} - \eta_{h,o}) = \\ \qquad = \Delta_h \{y + \sum_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota} e_{\iota,\ell}(y)\} + O(h^{\nu^*_{q_\ell}}) \end{cases}$$

for $p_\ell \leqq q_\ell$ and (4.20) for $p_\ell > q_\ell$. □

Instead of the original iteration method (4.7) we have studied in Theorem 4.3 the method defined by (4.10). What did we lose by this change?

Theorem 4.4: *Under the conditions of* Theorem 4.3 *one finds, using* (4.7) *instead of* (4.10), *again the relations* (4.19) *and* (4.20). *In* (4.19) *only the* $e_{\iota,\ell}$, *but not the* p_ℓ, q_ℓ *are to be changed. That means, from the viewpoint of asymptotic expansions: The relatively simple method* (4.10) *works as well as the relatively complicated method* (4.7).

Proof: Since we, usually, cannot solve $\Phi_h(\xi_{h,\ell-1}) = \Delta_h{}^o F(y_{\ell-1})$ in (4.7) directly, we do it via (4.8). Let us substitute $\Phi_h^*(\eta_{h,o})$ for $\Phi_h'(\eta_{h,o})$ in (4.8). In (4.8) we will perform only a few iterations. So the following argument is valid: Comparing (4.8) and (4.10) we find that $\zeta_o = \eta_{h,o}$ implies $\zeta_1 = \xi_{h,\ell-1}$ from (4.10). Now by (4.23)

$$\xi_{h,\ell-1} - \eta_{h,o} = \Delta_h\{ \sum_{\iota=p_{\ell-1}}^{p_\ell -1} h^{\nu_\iota} e_{\iota,\ell-1}(y) + \sum_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota} e^+_{\iota,\ell}(y)\} + O(h^{\nu^*_{q_\ell}}) .$$

To compute ζ_2 from (4.10) we have to determine $\Phi_h(\zeta_1)$ and we find with $\xi_{h,\ell-1} = \zeta_1$, $\Phi_h(\eta_{h,o})$ and the (ν_q,ν_p)-smoothness of $\mathfrak{M}$

$$\Phi_h(\zeta_1) = \Phi_h^{\langle 1\rangle}(\eta_{h,o})(\xi_{h,\ell-1}-\eta_{h,o})$$

$$+ \sum_{\sigma=2}^{[\nu_q/\nu_p]} \frac{1}{\sigma!} \Phi_{h(\nu_q,\nu_p)}^{(\sigma)}(\eta_{h,o})(\xi_{h,\ell-1}-\eta_{h,o})^{\sigma} + O(h^{\nu_{q_\ell}^*}) .$$

This and the conditions in Definition 4.1 imply

$$- \{\Phi_h(\zeta_1) - \Delta_h{}^{o}F(y_{\ell-1})\} = \Delta_h \{\sum_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota} g_{\iota,\ell}(y)\} + O(h^{\nu_{q_\ell}^*}) .$$

So again using (4.8) and the $\mathfrak{M}$-admissibility of $(\Phi_h^*(\eta_{h,o}), F^*(y_o))$ we find

$$(4.25) \qquad \zeta_2 - \zeta_1 = - \sum_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota} \tilde{g}_{\iota,\ell}(y) + O(h^{\nu_{q_\ell}^*})$$

where the $\tilde{g}_{\iota,\ell}(y) \neq e_{\iota,\ell}$, $\iota=p_\ell(1)q_\ell$, unless the $e_{\iota,\ell-1}$, $\iota=p_{\ell-1}(1)p_\ell-1$, are reproduced more exactly than we required and unless additional conditions on the higher derivatives are imposed. If we now define an improved $\eta_{h,\ell}^*$, analogously to (4.7) and (4.10) by

$$\eta_{h,\ell}^* := \eta_{h,\ell-1} - (\zeta_2-\eta_{h,o}) = \eta_{h,\ell-1} - (\zeta_2-\zeta_1)-(\zeta_1-\eta_{h,o}) = \eta_{h,\ell}-(\zeta_2-\zeta_1)$$

we find with (4.24), (4.25) and $\tilde{g}_{\iota,\ell}(y) \neq e_{\iota,\ell}(y)$

$$\eta_{h,\ell}^* = \Delta_h \{y + \sum_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota} e_{\iota,\ell}^*(y)\} + O(h^{\nu_{q_\ell}^*}) .$$

So $\eta^{*}_{h,\ell}$ has the same asymptotic expansion like $\eta_{h,\ell}$ and further iterations via (4.8) would give the same result. □

Again, like in 3., it is not necessary to know $F(y_{\ell-1})$ in (4.10) exactly, but only approximatively: If we use in (4.10) an approximation ψ^{*} to $F(y_{\ell-1})$ analogous to (3.17), we find

$$
(4.26)\quad \begin{cases}
\Phi_h(\eta^{*}_{h,o}) = 0, \text{ so } \eta^{*}_{h,o} = \eta_{h,o}\,, \\[2ex]
\Phi^{*}_h(\eta^{*}_{h,o})(\xi^{*}_{h,\ell-1} - \eta^{*}_{h,o}) = \psi^{*}(\eta^{*}_{h,\ell-1}) \text{ with} \\[2ex]
\psi^{*}(\eta^{*}_{h,\ell-1}) = \Delta^{o}_h\{F(T_\delta \eta^{*}_{h,\ell-1}) + \sum\limits_{\iota=p_\ell}^{q_\ell} h^{\nu_\iota} g_{\iota,\ell-1}(y)\} + O(h^{\nu^{*}_{q_\ell}}) \\[2ex]
\eta^{*}_{h,\ell} := \eta^{*}_{h,\ell-1} - (\xi^{*}_{h,\ell-1} - \eta_{h,o})\,.
\end{cases}
$$

<u>Theorem 4.5:</u> *Define* $\eta^{*}_{h,\ell}$ *by* (4.26) *and let the conditions of Theorem* 4.3 *be satisfied with* $\eta_{h,\ell-1}$, $e_{\iota,\ell-1}$ *in* (3.11) *resp.* (3.12) *replaced by* $\eta^{*}_{h,\ell-1}$, $e^{*}_{\iota,\ell-1}$. *Then* (4.19) *resp.* (4.20) *are valid with* $\eta_{h,\ell}$, $e_{\iota,\ell}$ *replaced by* $\eta^{*}_{h,\ell}$, $e^{*}_{\iota,\ell}$.

We now proceed to study the relations between the results in 3. and 4.. (3.9) and (4.10) look very alike and so the following Theorem is not astonishing:

Theorem 4.6: *Let $\mathfrak{M}$ be a linear discretization method and let* $F^*(y_o)$ *in* (3.9) *satisfy* (3.5) *and* $\phi_h(F^*(y_o)) = \Phi_h^*(\eta_{h,o}) + O(h^{\nu_{q+1}})$ *with the* $\Phi_h^*(\eta_{h,o})$ *in* (4.10) *resp.* (4.26) *and let* $\psi(\Delta_h u) = \psi^*(\Delta_h u) + O(h^{\nu_{q+1}})$. *If we write the results of* (3.9) *resp.* (3.16) *as* $\eta_{h,\ell}$ *resp.* $\eta^*_{h,\ell}$ *and the results of* (4.10) *resp.* (4.26) *as* $\bar{\eta}_{h,\ell}$ *resp.* $\bar{\eta}^*_{h,\ell}$, *then*

$$(4.27)\quad \begin{cases} \eta_{h,\ell} = \bar{\eta}_{h,\ell} + O(h^{\nu^*_{q_\ell}}) \quad , \quad \ell=0,1,\ldots \\ \\ \eta^*_{h,\ell} = \bar{\eta}^*_{h,\ell} + O(h^{\nu^*_{q_\ell}}) \quad , \quad \ell=0,1,\ldots \; . \end{cases}$$

Proof: With the denominations in Theorem 4.6 we have by (4.10) resp. (4.26)

$$\bar{\eta}_{h,\ell} - \bar{\eta}_{h,\ell-1} = -(\bar{\xi}_{h,\ell-1} - \bar{\eta}_{h,o}) \; ,$$

$$\xi_{h,\ell-1} = \bar{\xi}_{h,\ell-1}, \eta_{h,o} = \bar{\eta}_{h,o}$$

$$\bar{\eta}^*_{h,\ell} - \bar{\eta}^*_{h,\ell-1} = -(\bar{\xi}^*_{h,\ell-1} - \bar{\eta}_{h,o})$$

and therefore by (4.10) and $\phi_h(F^*(y_o)) = \Phi^*_h(\eta_{h,o}) + O(h^{\nu_{q+1}})$

$$- \Phi^*_h(\eta_{h,o})(\bar{\eta}_{h,\ell} - \bar{\eta}_{h,\ell-1}) = - \phi_h(F^*(y_o))(\bar{\eta}_{h,\ell} - \bar{\eta}_{h,\ell-1}) + O(h^{\nu_{q+1}})$$

$$= \Delta_h F(y_{\ell-1}) \; .$$

So, if (4.27) is already proved for $\ell-1$, we have

$$\phi_h(F^*(y_o))(\bar{\eta}_{h,\ell} - \eta_{h,\ell-1}) = -\Delta_h F(y_{\ell-1}) + O(h^{\nu_{q+1}}) \; ,$$

and so

$$\bar{\eta}_{h,\ell} = \eta_{h,\ell} + O(h^{\nu_{q+1}}) .$$

The results for $\eta^{**}_{h,\ell}$ are obtained in the same way. □

The idea of using N.P. goes back to ZADUNAISKY [13,14,15], again discussed by STETTER [12]. They treated initial value problems of ordinary differential equations where STETTER used our first method, too. The method of N.P. was applied by FRANK, HERTLING, UEBERHUBER [4,5,6] to initial and boundary value problems of ordinary differential equations. In the next papers we will give corresponding results for non-smooth starting values y_o and present some examples of our general theory.

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To solve $Fy = 0$ numerically we use two different methods, the first of which is sketched already in [3]. Secondly, we introduce a neighbouring problem (N.P.) $Fu = d$, $\|d\|$ small, with known solution. We solve the original problem and the N.P. with the "same discretization method". The known error of the N.P. is used as an estimation for the unknown error of the original problem. These procedures are used iteratively and their relations are discussed. In subsequent papers we will apply our general theory to some special cases and will discuss relations to collocation methods and to Pereyra's deferred correction methods [8,9].

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